International Journal of Theoretical Physics, Vol. 33, No. 9, 1994

Undecidable Hopf Bifurcation with Undecidable Fixed Point

N. C. A. da Costa[1] and F. A. Doria[2]

Received September 24, 1993

We exhibit a polynomial dynamical system where one cannot decide whether a Hopf bifurcation occurs. Therefore one cannot decide whether there will be parameter values such that a stable fixed point becomes an unstable one. Related incompleteness results for previously described axiomatized versions of dynamical systems theory are also discussed.

1. INTRODUCTION

In 1974 the American Mathematical Society organized a symposium to evaluate the state of the solution of Hilbert's problems of 1900. A new list of problems was drawn up at the symposium to represent the contemporary view about the main trends in mathematics. Arnold (1976) contributed the following queries to the portion that dealt with dynamical systems theory in that new list:

> *Is the stability problem for stationary points algorithmically decidable?* The well-known Lyapunov theorem solves the problem in the absence of eigenvalues with zero real parts. In more complicated cases, where the stability depends on higher order terms in the Taylor series, there exists no *algebraic* criterion.
> *Let a vector field be given by polynomials of a fixed degree, with rational coefficients. Does an algorithm exist, allowing to decide, whether the stationary point is stable?*
> A similar problem: Does there exist an algorithm to decide, whether a plane polynomial vector field has a limit cycle?

Dedicated to the memory of Leopoldo Nachbin (1922–1993), mathematician, mentor, and friend.

[1]Research Group on Logic and Foundations, Institute for Advanced Studies, University of São Paulo, 05655-010 São Paulo SP, Brazil.
[2]Research Center on Mathematical Theories of Communication, School of Communications, Federal University at Rio de Janeiro, 22295–900 Rio RJ, Brazil.

0020-7748/94/0900-1885$07.00/0

A more detailed statement of those problems was recently communicated to the authors by Arnold (personal communication, 1992):

> In my problem the coefficients of the polynomials of known degree and of a known number of variables are written on the tape of the standard Turing machine in the standard order and in the standard representation.
>
> The problem is whether there exists an algorithm (an additional text for the machine independent of the values of the coefficients) such that it solves the stability problem for the stationary point at the origin (i.e., always stops giving the answer "stable" or "unstable").
>
> I hope, this algorithm exists if the degree is one. It also exists when the dimension is one. My conjecture has always been that there is no algorithm for some sufficiently high degree and dimension, perhaps for dimension 3 and degree 3 or even 2. I am less certain about what happens in dimension 2.
>
> Of course the nonexistence of a general algorithm for a fixed dimension working for arbitrary degree or for a fixed degree working for an arbitrary dimension, or working for all polynomials with arbitrary degree and dimension would also be interesting.
>
> Integers were introduced in the formulation only to avoid the difficulty of explaining the way the data are written on the machine's tape. The more realistic formulation of the problem would require the definition of an analytic algorithm working with real numbers and functions (defined as symbols). The algorithm should permit arithmetical operations, modulus, differentiation, integration, solution of nondifferential equations (also for implicit functions in situations where conditions for the implicit function theorems are violated), exponentiation, logarithms, evaluation of 'computable' functions for 'computable' arguments.
>
> The conjecture is that with all those tools one is still unable:
>
> 1. To solve the general stability problem starting from the right hand side functions as symbols with which one may perform the preceding operations.
> 2. To solve the above problem for polynomial vectorfields with real or complex coefficients.
> 3. To solve them with integer coefficients.
>
> However as far as I know there are no words in logic to describe the above problem and I have thus preferred to stop at the level of algorithms in the usual sense rather than to try to explain to logicians the meaning of the impossibility of the solution of differential equations of a given type by quadratures (e.g., in the Liouville case in classical mechanics or in the theory of second order ordinary differential equations). The main difficulty here is that the solvability or unsolvability should be defined in a way that makes evident the invariance of this property under admissible changes of variables defined by functions that one can construct from the right hand side of the equations in a given coordinate system. In other terms we should explicitly describe the structure of the manifold where the vectorfield is given, with respect to which the equation is nonintegrable.
>
> In the usual approach this structure is a linear space structure, and I think it is too restrictive.
>
> In any case I would like to know whether you think you have proved my conjectures on polynomial vectorfields with integer coefficients:

- For some pair (degree, dimension);
- For some dimension;
- For some degree,

the polynomials being given on the tape of the machine in the standard form. If one of those undecidability conjectures is proved, it would be interesting to know for which pair (degree, dimension), or value of the dimension or value of the degree is the undecidability proven.

We give here an example related to those questions. Specifically, we exhibit a polynomial dynamical system where the origin is a fixed point but such that we cannot algorithmically decide whether a bifurcation related to the Hopf bifurcation occurs. That example leads us to the concluding question in Arnold's original formulation:

> A similar problem: Does there exist an algorithm to decide, whether a plane polynomial vector field has a limit cycle?

The solution should obey, according to Arnold, the following procedure:

> In my problem the coefficients of the polynomials of known degree and of a known number of variables are written on the tape of the standard Turing machine in the standard order and in the standard representation.
> The problem is whether there exists an algorithm (an additional text for the machine independent of the values of the coefficients) such that it solves the [required] problem.

[See above for the complete quote; a full discussion of the whole set of problems, together with the underlying mathematical machinery, appears elsewhere (da Costa *et al.* (1993*b*).]

We first show in the present paper that there is no general algorithm to decide, for an analytical dynamical system, whether a Hopf bifurcation occurs.

Second, we build out of the first example a dynamical system which is polynomial over Z and which turns out to be a particular perturbation of a set of harmonic oscillators. We use that system to prove that we cannot algorithmically decide whether there will be parameter values so that a stable fixed point at the origin will become an unstable one when the system is polynomial.

We also discuss related incompleteness phenomena.

2. MAIN TOOLS

Main references are two previous papers of ours (da Costa and Doria, 1991*a*; da Costa *et al.*, 1993*a*). We can think about our intuitive mathematical setting as a theory $T \supseteq \mathrm{ZFC}$, where ZFC denotes Zermelo–Fraenkel set theory plus (at least the countable) Axiom of Choice.

Notation is as usual. In particular ω are the natural numbers; Z are the integers, and R are the reals. We need a well-known result by Richardson (1968) as improved by Caviness (1970). If f is a function, the expression that represents f in our discussions is denoted $\ulcorner f \urcorner$. We consider the set of all expressions $\ulcorner \mathscr{F}_n \urcorner$ for real-valued elementary functions over R^n; the corresponding set of functions is denoted $\mathscr{F}_n$. We begin:

Definition 2.1. $g \in \mathscr{F}_n$ is a *dominating function* for $f \in \mathscr{F}_n$ if:

1. For all real $x_1, \ldots, x_n, g(x_1, \ldots, x_n) > 1$.
2. For all real $x_1, \ldots, x_n$ and all real $|\Delta_i| < 1$, $i = 1, \ldots, n$ then

$$g(x_1, \ldots, x_n) > f(x_1 + \Delta_1, \ldots, x_n + \Delta_n)$$

Lemma 2.2. If $f \in \mathscr{F}_n$, then we can explicitly and algorithmically construct an expression $\ulcorner g \urcorner$ that represents a dominating function g for f (da Costa and Doria, 1991*a*; Richardson, 1968).

Proposition 2.3. If $\ulcorner p(m, x_1, \ldots, x_n) \urcorner \in \ulcorner \mathscr{F}_n \urcorner$, $m \in \omega$, is a family of polynomial expressions over Z parametrized by m, then we can explicitly and algorithmically construct an expression $\ulcorner f \urcorner \in \ulcorner \mathscr{F}_n \urcorner$ for a function f such that:

1. For all $x_1, \ldots, x_n \in \omega$, $p(m, x_1, \ldots, x_n) \neq 0$ if and only if for all nonnegative $x_1 \ldots \in \mathsf{R}$, $f(m, x_1, \ldots, x_n) > 1$.
2. There are $x_1, \ldots, x_n \in \omega$ so that $p(m, x_1, \ldots, x_n) = 0$ if and only if there are nonnegative $x_1, \ldots, x_n \in \mathsf{R}$ so that $f(m, x_1, \ldots, x_n) = 0$ and if and only if there are nonnegative $x_1, \ldots, x_n \in \mathsf{R}$ so that $0 \leq f(m, x_1, \ldots, x_n) \leq 1$.

Proof. See Caviness (1970) and Richardson (1968). We give the explicit form for such an expression for f:

$$\ulcorner f(m, x_1, \ldots, x_n) \urcorner = (n+1)^2 \left[p^2(m, x_1, \ldots, x_n) + \sum_{i=1}^{n} \sin^2 \pi x_i g_i^2 (m, x_1, \ldots, x_n) \right]$$

Here the g_i are expressions for dominating functions of $(\partial/\partial x_i)p^2$. ■

Remark 2.4. The whole thing is pretty simple: f remains strictly greater than 1 on the positive side of R if and only if the Diophantine equation $p = 0$ has no roots over the naturals ω; f will drop to 0 if and only if $p = 0$ has at least one root.

We can also state:

Corollary 2.5. If $\ulcorner p(m, x_1, \ldots, x_n) \urcorner \in \ulcorner \mathscr{F}_n \urcorner$, $m \in \omega$, is a family of polynomial expressions over Z parametrized by m, then we can explicitly and algorithmically construct an expression $\ulcorner F \urcorner \in \ulcorner \mathscr{F}_n \urcorner$ for a function F such that:

1. For all $x_1, \ldots, x_n \in \omega$, $p(m, x_1, \ldots, x_n) \neq 0$ if and only if for all $x_1 \ldots \in \mathrm{R}$, $F(m, x_1, \ldots, x_n) > 1$.
2. There are $x_1, \ldots, x_n \in \omega$ so that $p(m, x_1, \ldots, x_n) = 0$ if and only if there are $x_1, \ldots, x_n \in \mathrm{R}$ so that $F(m, x_1, \ldots, x_n) = 0$ and if and only if there are $x_1, \ldots, x_n \in \mathrm{R}$ so that $0 \leq F(m, x_1, \ldots, x_n) \leq 1$.

Proof. See Caviness (1970) and Richardson (1968). Again we give the explicit form for such an expression for F:

$$\ulcorner F(m, x_1, \ldots, x_n) \urcorner = f(m, x_1^2, \ldots, x_n^2) \quad \blacksquare$$

Corollary 2.6. There is no general algorithm to decide, for an expression $\ulcorner h \urcorner \in \mathscr{F}_n$ and for any value of the variables of h, whether $h > 1$ or $h > 0$ (so that $h = 0$ sometimes).

Proof. Make p equal to a universal polynomial in Proposition 2.5. $\blacksquare$

The previous results lead to an explicit expression for the halting function in the theory of Turing machines. Richardson (1968) proves the following two results:

Corollary 2.7. We can explicitly and algorithmically construct an expression $\ulcorner c \urcorner$ for a function in $\ulcorner \mathscr{G}_n \urcorner = \ulcorner \mathscr{F}_n \urcorner \cup |\ldots|$, where $|\ldots|$ is the absolute value function ($\ulcorner \mathscr{G}_n \urcorner$ denotes the set of expressions for $\mathscr{F}_n$ to which we add $|\ldots|$ and close everything with respect to it), such that:

1. $p(m, x_1, \ldots, x_n) = 0$ has roots over ω if and only if there are real numbers so that $c(m, x_1, \ldots, x_n) > 0$.
2. $p(m, x_1, \ldots, x_n) = 0$ has no roots over ω if and only if, for all reals, $c(m, x_1, \ldots, x_n) = 0$.

Proof. Put

$$c(m, \mathbf{x}) = |f(m, \mathbf{x}) - 1| - (f(m, \mathbf{x}) - 1)$$

Here $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$. $\blacksquare$

Corollary 2.8. We can explicitly and algorithmically construct an expression $\ulcorner c \urcorner$ for a 1-variable function in $\ulcorner \mathscr{G}_1 \urcorner = \ulcorner \mathscr{F}_1 \urcorner \cup |\ldots|$ such that:

1. $p(m, x_1, \ldots, x_n) = 0$ has roots over ω if and only if there are real numbers x so that $c(m, x) > 0$.
2. $p(m, x_1, \ldots, x_n) = 0$ has no roots over ω if and only if, for all real x, $c(m, x) = 0$.

(Here we deal with a real-valued 1-variable function.)

Proposition 2.9 (The Halting Function). We can explicitly and algorithmically construct an expression $\theta(m, n)$ for the halting function in the theory of Turing machines within the language of classical elementary analysis. $\theta(m, n)$ satisfies:

1. $\theta(m, n) = 1$ if and only if the machine $M_m(n)$ halts.
2. $\theta(m, n) = 0$ if and only if the machine $M_m(n)$ never halts.

[$M_m(n)$ is the machine of index m that takes n as its input.]

Proof. One possible (multidimensional) representation for θ is given by

$$\theta(m) = \sigma(K(m))$$

where

$$K(m) = \int_{\mathbb{R}^n} c(m, \mathbf{x}) e^{-\mathbf{x}^2} d^n\mathbf{x}$$

$c(m, \mathbf{x})$ is given through Corollary 2.3, σ is the sign function, $\mathbf{x}^2 = x_1^2 + \cdots + x_n^2$, and $d^n\mathbf{x}$ is the volume element. To go from $\theta(m)$ to $\theta(m, n)$ it is enough to use the inverse of the usual pairing function τ. A one-dimensional representation for θ is given with the help of the integral $K(m) = \int_{\mathbb{R}} c(m, x) \exp(-x^2)\, dx$, where $c(m, x)$ comes from Corollary 2.8. For other representations for the halting function see da Costa and Doria (1991*a*).

Remark 2.10. Notice that our constructions for θ suggest that the proper setting for those functions is a separable Hilbert space.

3. AN UNDECIDABLE HOPF BIFURCATION

We must now carefully distinguish between the two levels where our discussion proceeds:

- *The intuitive, mathematical level.* When we deal with the actual properties of a given dynamical system, we argue with the tools available in the toolbox of everyday mathematics.
- *The formal, metamathematical level.* Those tools can be formalized within a classical set-theoretic first-order language. At the second level we argue about the formal system realized in that language; the formal system (or portions of it) will be the object of our discussions.

Undecidability and incompleteness proofs are given here.

3.1. The Intuitive Level

We start at the intuitive level.

For a review of the Hopf bifurcation see Hopf (1942), Howard (1975), Marsden and McCracken (1976), and Ruelle and Takens (1971). We deal here with a polynomial autonomous dynamical system with coefficients in Z over an adequate finite-dimensional manifold $\mathsf{R}^k \times M$.

Example 3.1. Let us be given the following infinite denumerable family $\Delta(m)$ of polynomial dynamical systems over Z, parametrized by μ_m, $m \in \omega$, and defined on an adequate $\mathsf{R}^k \times M$:

$$\frac{d\xi}{dt} = -\eta + \xi(\mu_m - \xi^2 - \eta^2)$$
$$\frac{d\eta}{dt} = \xi + \eta(\mu_m - \xi^2 - \eta^2) \tag{1}$$

$$\frac{dy_i}{dt} = cw_i z_i$$
$$\frac{dz_i}{dt} = -cw_i y_i \tag{2}$$

$$\frac{dx_i}{dt} = w_i$$
$$\frac{dw_i}{dt} = -x_i \tag{3}$$

$$\frac{dc}{dt} = 0. \tag{4}$$

($i = 1, 2, 3, \ldots, n$ and $k = 4n + 3$.) Here,

$$\mu_m(x_1, \ldots, x_n, y_1, \ldots, y_n) = -\left\{(n+1)^2\left[q^2(m, x_1, \ldots, x_n) + \sum_{i=1}^{n} y_i^2 g_i^2(m, x_1, \ldots, x_n)\right] - \frac{1}{2}\right\} \tag{5}$$

q will just be specified; however, see Proposition 2.3. g_i is the dominating polynomial for $(\partial/\partial x_i)q$ as in that same proposition (with respect to p).

We will specify below

$$q(m, \ldots, x_i^1, x_i^2, x_i^3, x_i^4, \ldots) = p\left(m, \ldots, 1 + \sum_{j=1}^{j=4} (x_i^j)^2, \ldots\right)$$

p will turn out to be an adequate Diophantine universal polynomial; q will have (if any) solutions over Z.

Finally, for notational simplicity we put $q = q(m, x_1, \ldots, x_n)$.

Relevant initial conditions will be explicitly added when required.

System $\Delta(m)$ is built out of three pieces: equations (1) are a classical example for the Hopf bifurcations; equations (2) and (3) describe harmonic oscillators, while the single trivial equation (4) introduces (with the help of an adequate initial condition) the constant π in the expression we gave for μ_m. We take $M = \{0\} \cup \{\pi\}$ as the range of the orbits in the last equation.

We state and prove the main result in this subsection:

Proposition 3.2. For system $\Delta(m)$:

1. The origin $0 \in \mathbb{R}^k \times M$ is an isolated critical point.
2. If $\mu_m < 0$, then the origin is asymptotically stable.
3. If $\mu_m > 0$, then the origin is unstable.

Proof. First assertion is immediate. For the second assertion, consider the Lyapunov function:

$$V = (1/2)\left[\xi^2 + \eta^2 + c^2 + \sum_i (y_i^2 + z_i^2 + x_i^2 + w_i^2)\right]$$

whereas,

$$\dot{V} = (\xi^2 + \eta^2)(\mu_m - \xi^2 - \eta^2)$$

Then, for $\dot{V} < 0$, as $\xi^2 + \eta^2 > 0$ as close to zero as we wish, we get $\mu_m \leq 0$. As μ_m may range over the negative reals, the system will satisfy $\dot{V} < 0$ near the origin for the strict inequality $\mu_m < 0$. Thus $\mu_m < 0$ is a sufficient condition for the stability of the system $\Delta(m)$.

We must now discuss the case $\mu_m > 0$.

First, a remark:

Remark 3.3. A phase portrait for $\Delta(m)$ is complicated. Sketchily, we have: depending on m, either μ_m ranges over $(-\infty, 0)$ or over $(-\infty, 1]$. The values of μ_m are determined by the oscillations of the x_i, which are always bounded. Then:

- If $\mu_m \in (-\infty, 0)$, then for a given set of initial conditions for the x_i the parameter μ_m will remain within a bounded interval I on the negative real line. The (projected) orbits of equations (1) will be smooth deformations of the orbits one has for a fixed value of μ_m inside I. That argument is valid for all x_i, so that there is no instabilizing of $\mathbf{0}$.
- However, if $\mu_m \in (-\infty, 1)$, there will be a set of initial conditions for the x_i so that $\mu_m \in J \subset (0, 1]$. Here again a continuity argument shows that there is an attracting circle and an unstable origin for that particular orbit, just as in the case of a constant $\mu_m \in J$.

Thus the origin is unstabilized.

We can argue about $\Delta(m)$ out of the simpler system $\Delta'(m)$ given by equations (1) together with equation (5) for μ_m. We recall the bifurcation properties of system (1):

Proposition 3.4. Given the system

$$\frac{d\xi}{dt} = -\eta + \xi(\mu_m - \xi^2 - \eta^2)$$
$$\frac{d\eta}{dt} = \xi + \eta(\mu_m - \xi^2 - \eta^2) \tag{6}$$

then:

1. For $\mu_m < 0$, the origin is a stable fixed point.
2. For $\mu_m > 0$, the origin becomes an unstable fixed point, while the circle $\xi^2 + \eta^2 = \mu_m$ is attracting and stable in a neighborhood of the origin of the combined phase space–parameter space $\mathbb{R}^2 \times \mathbb{R}$.

(See Howard, 1975; Marsden and McCracken, 1976.)

Remark 3.5. We now go from $\Delta'(m)$ to the full system $\Delta(m)$; we make explicit all the variables that determine the value of the bifurcation parameters μ_m. Only instead of simply giving the ranges of those variables, we present them through differential equations plus boundary conditions.

The oscillators (3) plus freely defined boundary conditions give the range of the x_i. Equation (4) allows us to introduce π into our equations, while equations (2) define the y_i as adequate trigonometric functions of the x_i. Therefore everything below equations (1) has to do with parameter space.

About system (2)–(3) we see that:

- The whole system (2)–(3) is neutrally stable.
- System (3) integrates to

$$x_i = A'_i \sin t + B'_i \cos t$$
$$w_i = A'_i \cos t - B'_i \sin t$$

- Systems (2)–(4) integrate to

$$y_i = A_i \sin cx_i + B_i \cos cx_i$$
$$z_i = A_i \cos cx_i - B_i \sin cx_i$$

Therefore, systems (2)–(3) essentially describe a synchronized periodic motion on $\mathbb{R}^{2n'}$. We can make rotations in $\langle x, w\rangle$ and $\langle y, z\rangle$ spaces so that the $B'_i = B_i = 0$ and

$$y_i = A_i \sin cx_i$$
$$z_i = A_i \cos cx_i$$

The crucial point here is the following: given all admissible initial conditions for (3), the x_i span the whole of $\langle x \rangle$ space as each one of them ranges over $[-A_i, A_i]$.

We can now state and prove the remaining assertion in Proposition 3.2:

Lemma 3.6. If $\mu_m > 0$ sometimes, then the origin in $\Delta(m)$ is unstable.

Proof. Consider system $\Delta'(m)$ for a fixed value $\mu_m^0 > 0$. Since the origin is unstable, given an adequately small open ball B around $0 \in R^2$ there will be a t_0 so that, for $t > t_0$, an orbit $\gamma'(\mu_m^0, t)$ for $\Delta'(m)$ escapes B.

Now γ' is the projection of an orbit γ for Δ. Choose the first t_0 outside B so that $\gamma'(\mu_m^0, t_0) = \gamma(\mu_m(t_0), t_0)$. Due to the synchronization of the x_i, if T is their common period, for all $r \in \omega$, $\gamma(\mu_m(t_0 + rT), t_0 + rT)$ will lie outside the open neighborhood of the origin $B \times B'$, where B' is a neighborbood of the origin in the remainder of the (product) phase space. ■

The preceding remarks and results lead to the following:

Proposition 3.7. If $\mu_m < 0$, then $\Delta(m)$ has a stable fixed point at the origin; if $\mu_m > 0$, then equations (1) undergo a bifurcation, and the origin is unstabilized for the whole system $\Delta(m)$.

3.2. The Metamathematical Level

The undecidability results go as follows, and arise out of the Matijasevič result (Davis, 1982). First, an undecidability result on system $\Delta'(m)$:

Proposition 3.8. If p is a universal Diophantine polynomial, then for the system given by $\Delta'(m)$ together with equation (5) and $y_i = \sin \pi x_i$, there is no general algorithm to decide whether a Hopf bifurcation occurs.

Specifically:

1. The set of values for m such that a Hopf bifurcation occurs is recursively enumerable.
2. The complementary set of values for m such that the origin remains stable is productive.

Proof. Immediate, from Corollary 2.5. [Essentially, we have substituted in $\Delta'(m)$ the single parameters μ_m for a k-parameter family $\mu_m(\ldots)$.] ■

Proposition 3.9. There is a set of boundary conditions so that there is no general algorithm to decide, for an arbitrary $m \in \omega$, whether the system (1) in $\Delta(m)$ (Example 3.1) goes through a bifurcation.

Proof. It will undergo a bifurcation if a parameter value $\mu_m > 0$ close to $\mu_m = 0$ is allowed in the system. In Proposition 2.3 we put

$$\mu_m = -\left(\lceil f(m) \rceil - \frac{1}{2}\right)$$

For $p(m, x_1, \ldots, x_n)$ as in that same proposition, we use Lagrange's theorem and substitute $x_i = 1 + p_i^2 + q_i^2 + r_i^2 + s_i^2$ (Davis, 1982). Now, if we take $p(m, \ldots)$ to be a universal polynomial, Richardson's result implies that:

- $\mu_m > 0$ if and only if there are natural numbers so that $p(m, \ldots) = 0$.
- $\mu_m < 0$ if and only if for no natural number $p(m, \ldots) = 0$.

Now consider $\Delta(m)$. We add the boundary condition $c(0) = \pi$ to equation (4). Then equations (4)–(3) define πx_i as the argument for the oscillator equations (2). If we now put $A_i = 1$, for all $i = 1, 2, \ldots, n$, then equations (2)–(3) define Richardson's transform as in Proposition 2.3. [We used Lagrange's theorem so that $p(m, \ldots)$ may be defined over the integers $\mathbb{Z}$, and μ_m over the full reals.]

Therefore the set $\{m: \Delta(m) \text{ bifurcates}\}$ is recursively enumerable, but has a nonrecursive complement. ■

Corollary 3.10. For the boundary conditions in the previous proposition, the set $\{m: \Delta(m) \text{ bifurcates}\}$ is creative, and the set $\{m: \Delta(m) \text{ does not bifurcate}\}$ is productive.

Or:

Corollary 3.11. There is a set of boundary conditions so that there is no general algorithm to compute, for an arbitrary $m \in \omega$, whether there are values of μ_m so that the fixed point at the origin in $\Delta(m)$ becomes unstable.

In other words:

Corollary 3.12. For the boundary conditions in the preceding examples, the set

$$\{m: \Delta(m) \text{ has a stable origin}\}$$

is productive, while the set

$$\{m: \Delta(m) \text{ has an unstable origin}\}$$

is creative.

Corollary 3.13. There is no index j for a Turing machine M_j such that M_j takes as its input the coefficients of $\Delta(m)$ according to some prearranged order and stops and prints 1 if the origin remains stable, and prints 0 if it can be unstabilized.

Proof. If M_j exists, then we can decide, for each set of boundary conditions in an arbitrary $\Delta(m)$, whether the bifurcation occurs in system (1). Therefore we would end up with the solution of the halting problem for $M_j(m)$. ■

3.3. Fermat's Conjecture and the Hopf Bifurcation

Recall that there is a polynomial over the integers that represents Fermat's conjecture; such a polynomial will have no solution at all over ω if Fermat's conjecture is true and will have solutions for each set of natural numbers $\langle x, y, z, m \rangle$ that falsifies that conjecture. [Recall that recently A. Wiles has claimed a proof of the Taniyama conjecture, which implies Fermat's (K. Ribet, personal communication, 1993; Stewart, 1993); however, the construction given here can be applied to any intractable number-theoretic problem.]

We are now going to exhibit explicitly one such polynomial. In order to do so, we need the Diophantine characterization for the exponential relation $v = u^k$, where v, u, and k are natural numbers:

Proposition 3.14. $v = u^k$ if and only if there are natural numbers $x_1, \ldots, x_{20}$ such that $p(u, k, v, x_1, \ldots, x_{20}) = 0$, where the polynomial p over the integers is given below:

$$\begin{aligned}
&p(u, k, v, x_1, \ldots, x_{20}) \\
&\quad = [x_1^2 - (x_2^2 - 1)x_3^2 - 1]^2 + [x_4^2 - (x_2^2 - 1)x_5^2 - 1]^2 \\
&\qquad + [x_6^2 - (x_7^2 - 1)x_8^2 - 1]^2 + (x_5 - x_9 x_3)^2 + [x_7 - (1 + 4x_{10}x_3)]^2 \\
&\qquad + [x_7 - (x_2 + x_{11}x_4)]^2 + [x_6 - (x_1 + x_{12}x_4)]^2 \\
&\qquad + [x_8 - k + 4(x_{13} - 1)x_3]^2 + [x_3 - (k + x_{14}) + 1]^2 \\
&\qquad + \{[x_1 - x_3(x_2 - u) - v]^2 - (x_{15} - 1)^2(2x_2 u - u^2 - 1)^2\}^2 \\
&\qquad + [v + x_{16} - (2x_2 u - u^2 - 1)]^2 \\
&\qquad + [x_{17} - (u + x_{18})]^2 + [x_{17} - (k + x_{19})]^2 \\
&\qquad + [x_2^2 - (x_{17}^2 - 1)(x_{17} - 1)^2 x_{20}^2 - 1]^2
\end{aligned}$$

Proof. See Davis (1982). ■

We now write down the Diophantine equation that represents Fermat's conjecture:

Proposition 3.15. Fermat's conjecture is equivalent to the formal sentence below:

$$\forall x, y, z, m \in \omega \neg \exists u, v, w, r_1, \ldots, s_1, \ldots, t_1, \ldots \in \omega$$
$$\{x, y, z > 1 \wedge m > 2 \wedge [p^2(x, m, u, r_1, \ldots, r_{20})$$
$$+ p^2(y, m, v, s_1, \ldots, s_{20})$$
$$+ p^2(z, m, w, t_1, \ldots, t_{20}) + (u + v - w)^2 = 0]\}$$

or, equivalently,

$$\forall i, j, k, n \in \omega \neg \exists x, y, z, m, u, v, w, r_1, \ldots, s_1, \ldots, t_1, \ldots \in \omega$$
$$\{[p^2(x, m, u, r_1, \ldots, r_{20}) + p^2(y, m, v, s_1, \ldots, s_{20})$$
$$+ p^2(z, m, w, t_1, \ldots, t_{20}) + (u + v - w)^2$$
$$+ (i + 2 - x)^2 + (j + 2 - y)^2 + (k + 2 - z)^2 + (n + 3 - m)^2 = 0]\}$$

where p is given in Proposition 3.14.

Proof. Notice that $(x + 2)^{m+3} + (y + 2)^{m+3} = (z + 2)^{m+3}$ is equivalent to

$$\exists u, v, w \in \omega u = x^m \wedge v = y^m \wedge w = z^m \wedge u + v = w$$
$$\wedge x = i + 2 \wedge y = j + 2 \wedge z = k + 2 \wedge m = n + 3$$

We then obtain the Diophantine equation that represents the above sentence, and add the quantifiers (Davis, 1982); the conditions on x, y, z, m avoid both the trivial solutions and the Pythagorean equation. ■

We denote $\Delta(x, y, z, m)$ the version of $\Delta(m)$ associated to the Fermat system, and state:

Proposition 3.16. There is a system $\Delta(x, y, z, m)$ such that there will be an algorithm to check whether Fermat's conjecture is true if and only if there is an algorithm to verify that the origin in $\Delta(x, y, z, m)$ is always stable, for every quadruple $\langle x, y, z, m \rangle$.

Proof. First, put the polynomial described in Proposition 3.15 in the transform given by Proposition 2.3. Then obtain the corresponding $\mu(x, y, z, m)$ for the system described in Example 3.1. ■

A detailed treatment of those relations between hard number-theoretic questions and dynamical systems can be found in da Costa *et al.* (1993*a*).

4. RELATED INCOMPLETENESS PHENOMENA

From here on we formalize dynamical systems theory inside an adequate axiomatic first-order theory T (think of $T \supseteq \mathrm{ZF}$ plus the countable Axiom of Choice). The actual axiomatization procedures are performed with the help of Suppes predicates and can be found in da Costa and Doria (1992*a*, 1993*a*).

We suppose that T has a model **M** where arithmetic is standard. Therefore T is consistent under that assumption. From here on, whenever we refer to "standard models" we will be talking about those.

We can prove about T:

Proposition 4.1. There is a Diophantine polynomial $p(x_1, \ldots, x_k)$ such that $\mathbf{M} \models$ "There are no natural numbers $x_1, \ldots, x_k$ such that $p = 0$," while $T \nvdash$ "There are no natural numbers $x_1, \ldots, x_k$ such that $p = 0$" or $T \nvdash$ "$\neg$(There are no natural numbers $x_1, \ldots, x_k$ such that $p = 0$)."

That result (Davis, 1982) derives from a similar theorem by Post ("Within an adequate consistent formalization for computation theory there is a Turing machine that never halts over a given input, but such that we cannot prove that fact within the formalization") and is related to the usual undecidability and incompleteness proofs in formalized arithmetic.

A corollary follows:

Corollary 4.2. There is an expression $f(x_1, \ldots, x_k) \in \ulcorner \mathcal{F}_n \urcorner \cup | \ldots |$ for a function in elementary analysis formalized inside T such that $\mathbf{M} \models$ "For all $x_1, \ldots, f(x_1, \ldots) > 1$," but such that T neither proves nor disproves that assertion.

Therefore:

Proposition 4.3. There is an $m_0 \in \omega$ such that $\mathbf{M} \models$ "$\Delta(m_0)$ does not undergo a Hopf bifurcation," but such that T can neither prove nor disprove that assertion.

4.1. Fermat's Conjecture Revisited

Our result on Fermat's conjecture and the nature of the fixed point for the corresponding system (Proposition 3.16) can be restated as:

Proposition 4.4. T is such that $T \vdash$ "Fermat's conjecture" if and only if, for all x, y, z, m, $T \vdash$ "The fixed point at the origin of $\Delta(x, y, z, m)$ is stable."

4.2. Incompleteness Implies Undecidability

We conclude with another proof of the nonexistence of an algorithm that decides whether a given fixed point is stable:

Proposition 4.5. There is no general algorithm to decide, for an arbitrary polynomial dynamical system of dimension $k \geq \dim \Delta(m)$ and of degree $d \geq \deg \mu_m + 1$, whether a fixed point is stable or not.

Sketch of Proof. Since T can be any extension of ZFC, and since any such theory satisfies Proposition 4.3, the existence of one such algorithm would be incompatible with that proposition. ■

Remark 4.6. Dimension k of $\Delta(m)$ ranges from 47 (out of a universal polynomial p of dimension equal to 11) coupled to a ridiculously high degree, to around 300 (for a universal polynomial p of dimension near 70) where the degree of the system is around 30.

A discussion of those and related undecidability and incompleteness results can be found in da Costa and Doria (1991*a*) and Stewart (1991).

5. CONCLUSION

We have here a full undecidability and incompleteness result for polynomial dynamical systems that might undergo a Hopf-related bifurcation; and we have shown that in the present situation incompleteness implies undecidability.

Several points should be raised here. The chief question has to do with the computability properties of continuous (nondiscrete) objects. Computation theory is a theory about discrete objects and the way they are handled in mathematics. Dynamical systems are continuous, and so standard wisdom suggests that in order to deal directly with their computability properties one should first extend our ideas about computability to continuous things. There are several possible approaches here; the oldest, for example, arises out of the definition of a "computable real" as an infinite computable sequence of digits.

Our approach is very different, and stems from ideas in algebraic computation. Essentially one notices that an equation $f(x_1, \ldots, x_n) = 0$ defined on R^n and built out of elementary functions (polynomials, sines, exponentials, and the like) has in general an infinite number of solutions, even if $n = 1$. Richardson's map allows us to establish a correspondence between polynomials and some objects built out of elementary functions; we thus translate all the undecidability properties of Diophantine equations into the language of elementary function theory, which is classical real analysis.

As in the case of algorithms, when we carefully distinguish between "algorithms" (computation procedures) and "algorithmic functions" (functions which are proved to have an algorithm), we must now separate "expressions" (which we handle in our computations) from the functions that they represent. As all mathematics is formally presented through expressions, we are here simply emphasizing a point that tends to become blurred in less rigorous treatments. We can 'visualize' mathematical objects in several shapes, forms, and intuitions, but when we settle down to prove things about them, those shapes and forms collapse into strings of letters, which are the only available data we concretely have about the mathematical universe.

What Pertains to Dynamical Systems?

Also, we have often heard that results such as the ones presented in this paper do not really belong to dynamical systems theory. Therefore we ask, what pertains to dynamical systems? "Strange attractors, stable manifolds, bifurcations of vectorfields," is the usual answer—a well-known zoo whose exhibits play no essential part in our proofs.

However, such an objection masks an epistemological prejudice and an unscientific belief. The prejudice is contained in the assertion, "a domain of knowledge is defined by the objects studied within that domain during a given historical period." It is enough to go through the history of physics to see the folly of such a conception; in mathematics a good example is the concept of "smoothness": smoothness (*bien sûr, avant la lettre*) was restricted to polynomials in the 17th and 18th centuries; then to analytic functions in the 19th century; then to C^∞ objects in the first half of the 20th century. Today one does nothing interesting about, say, spaces of metrics in general relativity if we do not add Sobolev spaces of functions to our collection of "smooth" functions.

The unscientific belief was the hope that weird gizmos such as strange attractors should somehow be related to weird results such as Gödel's incompleteness theorems. There are some quite interesting explorations about that theme (Tomita, 1984); the idea is, say, that the onset of turbulence in a dynamical system has a strict parallel in the 'onset' of incompleteness in arithmetic when one goes from Pressburger arithmetic to full-fledged arithmetic. Yet we can see here that undecidability and incompleteness may be present when one considers a simple Hopf bifurcation. No turbulence, no strange attractors.

The *Canard* and a Free Particle That Looks Chaotic

Now consider the following situation: let's go through the usual nonstandard description of the *canard* (Albeverio *et al.*, 1986, p. 33; see also Chuaqui, 1991) in a dynamical system. A nonstandard model may be seen as arising out of the following setting: we are given an axiomatic formulation for arithmetic which we then extend to our theory T, supposed consistent. We add to T an undecidable arithmetic statement; out of that statement we can concoct a Diophantine equation $p(x_1, \ldots, x_n)$ which has no roots *in any standard model for* T, but which has roots *in some nonstandard models*. We thus get our nonstandard model: it is one of the models where $p = 0$ for some (infinite) numbers.

Now again out of p we can obtain a function $\theta(p)$ such that $\theta = 1$ if p has roots, and $\theta = 0$ if p has no roots. Let X represent a single free particle over an adequate R^n and let Y represent a chaotic system on the same domain (da Costa and Doria, 1991*a*). Then

$$Z = \theta(p)X + (1 - \theta(p))Y$$

is chaotic in all standard models. *Yet it equals a single free particle in our nonstandard model!* We can go through the beautiful nonstandard description of the *canard* in that model. But when we simulate the expression for Z on a computer screen, we get a tangled system which will pass the usual statistical tests for randomness. For the elementary arithmetic portion of a standard model is recursive, so that it will be simulated on a computer screen.

Therefore consistency requires that when dealing with the nonstandard approach to physical systems we must be careful with the behavior of those undecidable statements of ours. They are essentially bifurcation points for theories, actually not very different from the bifurcation points in dynamical systems, as the construction described in this paper shows that a system may or may not undergo a phase transition, depending on the way one decides an undecidable statement.

Conclusion

Finally there is a remark attributed to S. Smale that results such as the present ones which show that undecidability and incompleteness are widespread in mathematics essentially imply that we must rethink our current views about the foundations of mathematics. Mathematics should not be blamed for them; the trouble lies in the way we have straitjacketed it into formalism.

Well, we fully agree with that remark. We go even further: a constant rethinking of the foundations of mathematics should be the rule. It is as if there could not be stable foundational views for mathematics; as if an everlasting shadow were always to protect the abyss.

ACKNOWLEDGMENTS

The results presented here were originally conceived when F.A.D. was visiting the IMSSS at Stanford University as a Senior Fulbright Scholar in 1990. This paper was partially supported by FAPESP and CNPq (Brazil) and JNICT (Portugal). The authors thank their institutions for continuing support of their work. Special thanks are due to Prof. J. H. Vilhena de Paiva, Dean for Humanities at Rio's Federal University, and to Prof. C. A. Messeder, Director of the School of Communications at the same university, for enthusiastic support of the activities of the Research Center on Mathematical Theories of Communication. F.A.D. would also like to thank J. Conboy, P. Ferrand, S. Jesus, R. Potting, P. Relvas, C. Vaz, and W. Vaz, members of the Math-Phys seminar for the fall term 1992 at the University of Algarve, Portugal, where those ideas were presented and discussed.

Finally, formatting of this text is due to *Project Griffo* at the School of Communications, whose help is gratefully acknowledged.

REFERENCES

Albeverio, S., Fenstad, J. E., Hoegh-Krohn, R., and Lindstrom, T. (1986). *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press.

Arnold, V. I. (1976). Problems of present day mathematics, XVII (Dynamical systems and differential equations), *Proceedings Symposium on Pure Mathematics*, **28**, 59.

Caviness, B. F. (1970). *Journal of the ACM*, **17**, 385.

Chuaqui, R. (1991). *Truth, Possibility and Probability: New Logical Foundations of Probability and Statistical Inference*, North-Holland, Amsterdam.

da Costa, N. C. A., and Doria, F. A. (1991*a*). *International Journal of Theoretical Physics*, **30**, 1041.

da Costa, N. C. A., and Doria, F. A. (1991*b*). *Foundation of Physics Letters*, **4**, 363.

da Costa, N. C. A., and Doria, F. A. (1992*a*). Suppes predicates for classical physics, in *The Space of Mathematics*, J. Echeverría *et al.*, eds., de Gruyter, Berlin.

da Costa, N. C. A., and Doria, F. A. (1992*b*). Structures, Suppes predicates and Boolean-valued models in physics, in *Festschrift in Honor of V. I. Smirnov on his 60th Birthday*, P. Bystrov and J. Hintikka, eds., Kluwer.

da Costa, N. C. A., and Doria, F. A. (1992*c*). *Philosophica*, **50**, 901.

da Costa, N. C. A., and Doria, F. A. (1993*a*). Suppes predicates and the construction of unsolvable problems in the axiomatized sciences, in *Patrick Suppes, Mathematician, Philosopher*, P. Humphreys, ed., Kluwer.

da Costa, N. C. A., and Doria, F. A. (1993*b*). On Arnold's Hilbert Symposium problems, in *Proceedings of the 1993 Kurt Gödel Colloquium: Computational Logic and Proof Theory*, G. Gottlob, A. Leitsch, and D. Mundici, eds., Springer, pp. 152–158.

da Costa, N. C. A., Doria, F. A., and Furtado-do-Amaral, A. F. (1993*a*). *International Journal of Theoretical Physics*, **32**, 2187–2206.

da Costa, N. C. A., Doria, F. A., Baêta Segundo, J. A., and Krause, D. (1993*b*). On Arnol'd's 1974 Hilbert Symposium problems, to appear.

Davis, M. (1982). Hilbert's tenth problem is unsolvable, in *Computability and Unsolvability*, 2nd ed., Dover, New York.

Hopf, E. (1942). *Sachsische Akademie der Wissenschaften Mathematische-Physikalische Serie*, **94**, 1.

Howard, L. N. (1975). *Advances in Mathematics*, **16**, 246.

Marsden, J. E., and McCracken, M. (1976). *The Hopf Bifurcation and its Applications*, Springer.

Richardson, D. (1968). *Journal of Symbolic Logic*, **33**, 514.

Ruelle, D., and Takens, F. (1971). *Communications in Mathematical Physics*, **20**, 167.

Stewart, I. (1991). *Nature*, **352**, 664.

Stewart, I. (1993). *New Scientist*, **1993** (3 July), 14.

Tomita, K. (1984). Coarse-graining revisited: The case of macroscopic chaos, in *Chaos and Statistical Methods*, Y. Kuramoto, ed., Springer.